B.Math. (Hons.) Ist year
First semestral exam
January 5, 2022
Number Theory
Instructor : B.Sury
Answer ALL SIX questions.
Be BRIEF!

Q 1.
(i) Prove $\sum_{k=0}^{n} k\binom{n}{k}^2 = \binom{2n-1}{n-1}n$.
(ii) Prove that among 10 consecutive positive integers, at least one is co-prime to the product of the other 9.
Hint. For (i), either use counting or induction. For (ii), use the pigeon-hole principle.

OR

(a) Show that a positive integer n is a difference of two perfect squares if, and only if, it is NOT of the form $4k + 2$.
(b) If $n > 2$, prove that there exists a prime p with $n < p < n!$.
Hint for (b). Look at $n! - 1$.

Q 2. Find all the primes p which divides a number of the form $a^2 - 7$.
Hint. Use the quadratic reciprocity law.

OR

Show that there are infinitely many natural numbers n such that $n! - 1$ has at least two different prime factors.

OR

Suppose $n > 1$ is a Carmichael number (that is, $a^{n-1} \equiv 1 \bmod n$ for all $(a, n) = 1$). Show that n is square-free, and $(p - 1)|(n - 1)$ for each prime $p|n$.

Q 3. Let g be a multiplicative function and $f(n) = \sum_{d|n} g(d)$. Consider the $n \times n$ matrix A where $a_{ij} = f(GCD(i,j))$. Show that det $A = g(1)g(2)\cdots g(n)$.
Hint: Look at the matrix B with $b_{ij} = \sqrt{g(j)}$ when $j|i$ and $b_{ij} = 0$ if not. Relate A to B.

OR

Prove that there are infinitely many primes of the form $6n+1$.
Hint. If $p_1, \cdots, p_r$ are primes of the form $6k+1$, consider $(2p_1\cdots p_r)^2 + 3$ and use quadratic reciprocity law.

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Q 4. Let $p > 2$ be a prime and suppose a is a primitive root mod p. Prove that either a or $a+p$ is a primitive root mod p^n for all $n > 1$.

OR

Let p be a prime. Prove that the sequence $1^1, 2^2, 3^3, \cdots$ is periodic mod p with least period $p(p-1)$.

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Q 5. Prove:
(i) $\sum_{d^2|n} \mu(d) = \mu(n)^2$.
(ii) $\sigma(n)$ is odd if, and only if, $n = 2^k m^2$ with $k = 0$ or 1.

OR

Consider the von Mangoldt function Λ defined as $\Lambda(p^r) = \log p$ if p is a prime and $r > 0$ and as 0 otherwise. Prove $\Lambda(n) = -\sum_{d|n} \mu(d)\log(d)$.

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Q 6. If m, n are coprime positive integers, prove that

$$\sum_{i=1}^{[m/2]} [in/a] + \sum_{j=1}^{[n/2]} [jm/n] = [m/2][n/2].$$

OR

If p is an odd prime and $(ab,p) = 1$, show that the number of solutions (x,y) of the congruence $ax^2 + by^2 \equiv 1$ mod p is $p - \left(\frac{-ab}{p}\right)$.